

Stochastic Optimal Control with Delay in the Control II: Verification Theorem and Optimal Feedbacks

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Abstract

We consider a stochastic optimal control problem governed by a stochastic differential equation with delay in the control. Using a result of existence and uniqueness of a sufficiently regular mild solution of the associated Hamilton-Jacobi-Bellman (HJB) equation, see the companion paper [24], we solve the control problem by proving a Verification Theorem and the existence of optimal feedback controls.

Key words:

Optimal control of stochastic delay equations; Delay in the control; Lack of the structure condition; Second order Hamilton-Jacobi-Bellman equations in infinite dimension; Verification Theorem; Optimal Feedbacks; \mathcal{K} -convergence.

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1 Introduction

The aim of this paper is to solve, through the dynamic programming approach, a class of stochastic optimal control problems with delay in the control.

Stochastic optimal control problems governed by delay equations with delay in the control are usually harder to study than the ones when the delay appears only in the state (see e.g. [2, Chapter 4] in the deterministic case and [22, 23] in the stochastic case). When one tries to apply the dynamic programming method the main difficulty is the fact that, even in the simplified setting introduced first by Vinter and Kwong [31] in the deterministic case (see e.g. [22] for the stochastic case), the associated HJB equation is an infinite dimensional second order semilinear PDE that does not satisfy the so-called “structure condition”, which substantially means that the control can act on the system modifying its dynamics at most along the same directions along which the noise acts. The absence of such condition, together with the lack of smoothing properties which is a common feature of problems with delay, prevents the use of the known techniques, based on BSDE’s (see e.g. [17]) or on fixed point theorems in spaces of continuous functions (see e.g. [4, 5, 10, 20, 21]) or in Gauss-Sobolev spaces (see e.g. [8, 19]), to prove the existence of regular solutions of this HJB equation.¹

In the companion paper [24] we proved, using an ad hoc method based on the partial smoothing properties of the Ornstein-Uhlenbeck semigroup, that the HJB equation associated to the class of problems under study here, admits a unique *mild solution*, i.e. a solution which possesses enough regularity to give sense to the “candidate optimal feedback map” which depends on a suitable directional derivative (denoted by ∇^B) of the solution.

In this paper we start from such a result and exploit it to solve our class of problems. More precisely we prove here:

- (A) an approximation result for the solutions of the HJB equation, i.e. that the mild solutions can be approximated by classical solutions, to which Ito’s formula applies (Lemma 4.3);
- (B) a verification theorem for the control problem (Theorem 5.3);
- (C) the existence, under further assumptions on the Hamiltonian, of optimal controls in feedback form (Theorem 5.7).

These results allows to treat satisfactorily a specific class of state equations and data which arises naturally in many applied problems (see e.g. [13, 14, 22, 23, 26, 3]).

¹The viscosity solution technique can still be used (see e.g. [23]) but to prove existence (and possibly uniqueness) of solutions that are merely continuous.

The three points outlined above are the ones followed, in a different context, in [20]. However the specific features of our problems prevents the use of the same techniques. In particular the approximation result, which is a key tool here, must be completely reworked, as we explain in Remarks 4.4. and 5.2.

We finally note that, similarly to what often happens in the literature (see e.g. [14, 22, 23]) here we treat the case of “distributed delay” which gives rise to a bounded control operator in the state equation. The case of “pointwise delay”, even if it seems treatable with our approach, is left for future extensions of our research.

1.1 Our results in a simple motivating case

To be more clear we now briefly describe our setting and our main result in a special case. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and consider the following linear controlled Stochastic Differential Equation (SDE) in \mathbb{R} :

$$\begin{cases} dy(s) = a_0 y(s) ds + b_0 u(s) ds + \int_{-d}^0 b_1(\xi) u(s + \xi) d\xi + \sigma dW_s, & s \in [t, T] \\ y(t) = y_0, \\ u(\xi) = u_0(\xi), & \xi \in [-d, 0]. \end{cases} \quad (1.1)$$

Here W is a standard Brownian motion in \mathbb{R} , and $(\mathcal{F}_t)_{t \geq 0}$ is the augmented filtration generated by W . We assume $a_0, b_0 \in \mathbb{R}$, $\sigma > 0$. The parameter $d > 0$ represents the maximum delay the control takes to affect the system while b_1 is the density function taking account the aftereffect of the control on the system. The case treated here is when $b_1 \in L^2([-d, 0], \mathbb{R})$ (“distributed delay”) while a more difficult case which we leave for further research is when b_1 is a measure, e.g. a Dirac delta in $-d$ (“pointwise delay”).

The initial data are the initial state y_0 and the past history u_0 of the control. The control u belongs to $L^2_{\mathcal{F}}(\Omega \times [0, T], U)$, the space of predictable square integrable processes with values in $U \subseteq \mathbb{R}$, closed.

Such kind of equations is used e.g. to model the effect of advertising on the sales of a product (see e.g. [22, 23]), the effect of investments on growth (see e.g. [13] in a deterministic case), or, in a more general setting, to model optimal portfolio problems with execution delay, (see e.g. [3]) or to model the interaction of drugs with tumor cells (see e.g. [26] p.17 in the deterministic case).

In many applied cases (like the ones quoted above) the goal of the problem is to minimize the following objective functional

$$J(t, x_0, u_0; u(\cdot)) = \mathbb{E} \int_t^T (\bar{\ell}_0(s, y(s)) + \bar{\ell}_1(u(s))) ds + \mathbb{E} \bar{\phi}(y(T)). \quad (1.2)$$

where $\bar{\ell}_0 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $\bar{\ell}_1 : U \rightarrow \mathbb{R}$ and $\bar{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions satisfying suitable assumptions that will be introduced in Section 2.2. It is important to note that here $\bar{\ell}_0$, $\bar{\ell}_1$ and $\bar{\phi}$ do not depend on the past of the state and/or control. This is a very common feature of such applied problems.

A standard way² to approach these delayed control problems, introduced in [31] for the deterministic case (see [22] for the stochastic case) is to reformulate the above linear delay equation as a linear SDE in the Hilbert space $\mathcal{H} := \mathbb{R} \times L^2([-d, 0], \mathbb{R})$, with state variable $Y = (Y_0, Y_1)$ as follows.

$$\begin{cases} dY(s) = AY(s)ds + Bu(s)ds + GdW_s, & s \in [t, T] \\ Y(t) = x = (x_0, x_1), \end{cases} \quad (1.3)$$

where A generates a C_0 -semigroup (see next section for precise definitions) while, at least formally,³

$$B : \mathbb{R} \rightarrow \mathcal{H}, \quad Bu = (b_0 u, b_1(\cdot)u), \quad u \in \mathbb{R}, \quad G : \mathbb{R} \rightarrow \mathcal{H}, \quad Gx = (\sigma x, 0), \quad x \in \mathbb{R}. \quad (1.4)$$

Moreover $x_0 = y_0$ while $x_1(\xi) = \int_{-d}^{\xi} b_1(\varsigma) u_0(\varsigma - \xi) d\varsigma$, ($\xi \in [-d, 0]$) i.e. the infinite dimensional datum x_1 depends on the past of the control.

²It must be noted that, under suitable restrictions on the data, one can treat optimal control problems with delay in the control also by a direct approach without transforming them in infinite dimensional problems. However in the stochastic case such direct approach seems limited to a very special class of cases (see e.g. [27]) which does not include our model and models commonly used in applications like the ones just quoted.

³Note that, when b_1 is a measure, the above operator B is not bounded and this makes the problem more difficult.

The value function is defined as

$$V(t, x) := \inf_{u(\cdot) \in L^2_{\mathcal{F}}(\Omega \times [0, T], U)} \mathbb{E} \left(\int_t^T [\bar{\ell}(s, Y_0(s)) + \bar{\ell}_1(u(s))] ds + \bar{\phi}(Y_0(T)) \right)$$

The associated HJB equation (whose candidate solution is the value function) is

$$\begin{cases} -\frac{\partial v(t, x)}{\partial t} = \frac{1}{2} \text{Tr} \, GG^* \nabla^2 v(t, x) + \langle Ax, \nabla v(t, x) \rangle_{\mathcal{H}} + \bar{H}_{min}(\nabla v(t, x)) + \bar{\ell}_0(t, x), & t \in [0, T], x \in \mathcal{H}, \\ v(T, x) = \bar{\phi}(x_0), \end{cases} \quad (1.5)$$

where, defining the current value Hamiltonian \bar{H}_{CV} as

$$\bar{H}_{CV}(p; u) := \langle p, Bu \rangle_{\mathcal{H}} + \bar{\ell}_1(u) = \langle B^* p, u \rangle_{\mathbb{R}} + \bar{\ell}_1(u)$$

we have

$$\bar{H}_{min}(p) := \inf_{u \in U} \bar{H}_{CV}(p; u). \quad (1.6)$$

It is well known, see e.g. [32] Section 5.5.1, that, if the value function V is smooth enough, and if the current value Hamiltonian \bar{H}_{CV} always admits at least a minimum point, a natural candidate optimal feedback map is given by $(t, x) \mapsto u^*(t, x)$ where $u^*(t, x)$ satisfies

$$\langle \nabla V(t, x), Bu^*(t, x) \rangle_{\mathbb{R}} + \bar{\ell}_1(u^*(t, x)) = \bar{H}_{min}(\nabla V(t, x)).$$

i.e. where $u^*(t, x)$ is a minimum point of the function $u \mapsto \bar{H}_{CV}(\nabla v(t, x); u)$, $\mathbb{R} \rightarrow \mathbb{R}$. To take account of the presence of B it is convenient to define, for $z \in \mathbb{R}$,

$$H_{min}(z) := \inf_{u \in U} \{ \langle z, u \rangle_{\mathbb{R}} + \bar{\ell}_1(u) \} =: \inf_{u \in U} H_{CV}(z; u) \quad (1.7)$$

so that

$$\bar{H}_{min}(p) = \inf_{u \in U} \bar{H}_{CV}(p; u) = \inf_{u \in U} \{ \langle B^* p, u \rangle_{\mathbb{R}} + \bar{\ell}_1(u) \} =: H_{min}(B^* p). \quad (1.8)$$

From now on we will use H_{CV} and H_{min} in place of \bar{H}_{CV} and \bar{H}_{min} writing $H_{min}(\nabla^B v(t, x))$ in place of $\bar{H}_{min}(\nabla v(t, x))$ in (1.5).

Since $u^*(t, x)$ is a minimum point of the function $u \mapsto H_{CV}(B^* \nabla v(t, x); u)$, $\mathbb{R} \rightarrow \mathbb{R}$, the minimal regularity required to give sense to such term is the existence of $B^* \nabla v(t, x)$ which we will call $\nabla^B v(t, x)$ according to the definition and notation used e.g. in [18, 28] and in the companion paper [24]. In [24] a result of existence and uniqueness of mild solutions for such equation has been proved. Mild solution (defined through an integral form of (1.5)) are continuous and such that $\nabla^B v$ is well defined and continuous, hence the natural candidate optimal feedback map $u^*(t, x)$ above is well defined.

Notice that if the controlled state equation satisfies the “structure condition”, meaning that the control affects the system only through the noise, then by the so called BSDEs approach, see e.g. [17], the fundamental relation and the consequent verification theorem, can be proved also by applying the Girsanov Theorem. In our case the “structure condition” does not hold, since it would mean that $\text{Im} B \subseteq \text{Im} G$. This is an intrinsic feature of control problems with delay in the control since the fact that $\text{Im} B$ is not contained in $\text{Im} G$ is just due to the presence of the delay in the control. If the delay in the control disappears, then the structure condition holds, even if delay in the state is present (see e.g. [22, 23, 17, 29]).

Then solve the problem, as recalled in the beginning of this introduction, one needs to accomplish the steps (A)-(B)-(C), that we briefly introduce:

- (A) In Lemma 4.3 we show that if we suitably approximate the coefficients of the HJB equation (1.5), we obtain a sequence of functions $(w_n)_n$ which are strict solutions of the approximating HJB equations, and which are once differentiable in time and twice differentiable with respect to x . Moreover the sequence $(w_n)_n$ converges to v , the mild solution of the HJB equation, in the sense of the \mathcal{K} -convergence, see Definition 4.1.
- (B) In Theorem 5.3 we apply the fundamental relation proved in Proposition 5.1: in order to prove the fundamental relation it is crucial to apply the Ito formula, and this can be done thanks to the approximation performed in (A) and to a further approximation of the state.

- (C) In Theorem 5.7, under further regularity assumptions stated in Hypothesis 2.5, we solve the closed loop equation and so we show the existence of optimal controls in feedback form and the fact that the value function coincides with the solution of the HJB equation, see Theorem 5.8.

Finally notice that in the present paper we deal with a finite dimensional control delay equation (1.1), that here in the introduction we have presented in dimension one for the sake of simplicity. The same arguments apply if we consider the case of a controlled stochastic differential equation in an infinite dimensional Hilbert space \mathcal{H}_0 with delay in the control as follows.

$$\begin{cases} dy(t) = A_0 y(t)dt + B_0 u(t)dt + \int_{-d}^0 B_1(\xi)u(t+\xi)d\xi + \sigma dW_t, & t \in [0, T] \\ y(0) = y_0, \\ u(\xi) = u_0(\xi), & \xi \in [-d, 0). \end{cases} \quad (1.9)$$

Here W is a cylindrical Wiener process in another Hilbert space Ξ , A_0 is the generator of a strongly continuous semigroup in \mathcal{H}_0 , $\sigma \in \mathcal{L}(\Xi, \mathcal{H}_0)$, and we have to assume some smoothing properties for the Ornstein Uhlenbeck transition semigroup with drift term given by A_0 and diffusion equal to σ , see Remarks 3.2, 4.7 and 4.12 of [24] for more details.

1.2 Plan of the paper

In the first two Sections we present the control problem we treat and we recall some results on the state equation and on the HJB equation proved in [24], in the last two Sections we solve the control problem. In details, the plan of the paper is the following:

- in Section 2 we give some notations and we present the problem and the main assumptions;
- in Section 3 we collect some results proved in [24] and fundamental in solving the control problem: the partial smoothing property for the Ornstein-Uhlenbeck transition semigroup related to the state equations, and results on the existence of a mild solution of the HJB equation;
- in Section 4 we prove that this mild solution can be approximated by means of a sequence of strong solutions;
- Section 5 is devoted to solve the optimal control problem. In Subsection 5.1 we prove a verification theorem, and finally in Subsection 5.2 we identify the value function of the control problem with the solution of the HJB equation and we characterize the optimal control by a feedback law.

2 Some preliminary results on the control problem

In this section we collect and synthesize, for the reader's convenience, the basic material on our control problem which has been already exposed in the companion paper [24].

2.1 Notation and C -derivatives

Let H, K be real and separable Hilbert spaces, with norms given by $|x|_H$ and by $|x|_K$, respectively, or by $|x|$, if no confusion is possible, and by scalar product $\langle \cdot, \cdot \rangle_H$, $\langle \cdot, \cdot \rangle_K$, respectively, or simply by $\langle \cdot, \cdot \rangle$. The space $\mathcal{L}(H, K)$ denotes the space of bounded linear operators from H to K endowed with the usual operator norm.

In the following, by $(\Omega, \mathcal{F}, \mathbb{P})$ we denote a complete probability space, and by $L^2_{\mathcal{P}}(\Omega \times [0, T], H)$ the Hilbert space of all predictable processes $(Z_t)_{t \in [0, T]}$ with values in H , normed by $\|Z\|_{L^2_{\mathcal{P}}(\Omega \times [0, T], H)}^2 = \mathbb{E} \int_0^T |Z_t|^2 dt$.

Next we introduce some spaces of functions. We let H and Z be Hilbert spaces. By $B_b(H, Z)$ (respectively $C_b(H, Z)$, $UC_b(H, Z)$) we denote the space of all functions $f : H \rightarrow Z$ which are Borel measurable and bounded (respectively continuous and bounded, uniformly continuous and bounded).

Given an interval $I \subseteq \mathbb{R}$ we denote by $C(I \times H, Z)$ (respectively $C_b(I \times H, Z)$) the space of all functions $f : I \times H \rightarrow Z$ which are continuous (respectively continuous and bounded). $C^{0,1}(I \times H, Z)$ is the space of functions $f \in C(I \times H)$ such that for all $t \in I$ $f(t, \cdot)$ is Fréchet differentiable. By $UC_b^{1,2}(I \times H, Z)$

we denote the linear space of the mappings $f : I \times H \rightarrow Z$ which are uniformly continuous and bounded together with their first time derivative f_t and its first and second space derivatives $\nabla f, \nabla^2 f$. If $Z = \mathbb{R}$ we omit it in all the above spaces.

Next, to introduce some spaces of functions which are differentiable in suitable directions, we recall the definition of C -directional derivatives given in [28], Section 2, and in [18].

Definition 2.1 *Let H, K, Z be real Hilbert spaces. Let $C : K \rightarrow H$ be a bounded linear operator and let $f : H \rightarrow Z$.*

- *The C -directional derivative ∇^C at a point $x \in H$ in the direction $k \in K$ is defined as:*

$$\nabla^C f(x; k) = \lim_{s \rightarrow 0} \frac{f(x + sCk) - f(x)}{s}, \quad s \in \mathbb{R}, \quad (2.1)$$

provided that the limit exists.

- *We say that a continuous function f is C -Gâteaux differentiable at a point $x \in H$ if f admits the C -directional derivative in every direction $k \in K$ and there exists a linear operator, called the C -Gâteaux differential, $\nabla^C f(x) \in \mathcal{L}(K, Z)$, such that $\nabla^C f(x; k) = \nabla^C f(x)k$ for $x \in H, k \in K$. The function f is C -Gâteaux differentiable on H if it is C -Gâteaux differentiable at every point $x \in H$.*
- *We say that f is C -Fréchet differentiable at a point $x \in H$ if it is C -Gâteaux differentiable and if the limit in (2.1) is uniform for k in the unit ball of K . In this case we call $\nabla^C f(x)$ the C -Fréchet derivative (or simply the C -derivative) of f at x . We say that f is C -Fréchet differentiable on H if it is C -Fréchet differentiable at every point $x \in H$.*

Note that, in doing the C -derivative, one considers only the directions in H selected in the image of C . When $Z = \mathbb{R}$ we have $\nabla^C f(x) \in K^*$. Usually we will identify K with its dual K^* so $\nabla^C f(x)$ will be treated as an element of K .

If $f : H \rightarrow \mathbb{R}$ is Gâteaux (Fréchet) differentiable on H we have that, given any C as in the definition above, f is C -Gâteaux (Fréchet) differentiable on H and

$$\langle \nabla^C f(x), k \rangle_K = \langle \nabla f(x), Ck \rangle_H$$

i.e. the C -directional derivative is just the usual directional derivative at a point $x \in H$ in direction $Ck \in H$. Anyway the C -derivative, as defined above, allows us to deal also with functions that are not Gâteaux differentiable in every direction.

Next we introduce some spaces of functions, see also [24], Section 2.2.

Now we define suitable spaces of C -differentiable functions.

Definition 2.2 *Let I be an interval in \mathbb{R} and let H, K and Z be suitable real Hilbert spaces.*

- *We call $C_b^{1,C}(H, Z)$ the space of all functions $f : H \rightarrow Z$ which admit continuous and bounded C -Fréchet derivative. Moreover we call $C_b^{0,1,C}(I \times H, Z)$ the space of functions $f : I \times H \rightarrow Z$ belonging to $C_b(I \times H, Z)$ and such that, for every $t \in I$, $f(t, \cdot) \in C_b^{1,C}(H, Z)$. When $Z = \mathbb{R}$ we omit it.*
- *We call $C_b^{2,C}(H, Z)$ the space of all functions f in $C_b^1(H, Z)$ which admit continuous and bounded directional second order derivative $\nabla^C \nabla f$; by $C_b^{0,2,C}(I \times H, Z)$ we denote the space of functions $f \in C_b(I \times H, Z)$ such that for every $t \in I$, $f(t, \cdot) \in C_b^{2,C}(H, Z)$. When $Z = \mathbb{R}$ we omit it.*
- *Here and in the following point of the Definition we take $Z = \mathbb{R}$. For any $\alpha \in (0, 1)$ and $T > 0$ (this time I is equal to $[0, T]$) we denote by $C_\alpha^{0,1}([0, T] \times H)$ the space of functions $f \in C_b([0, T] \times H) \cap C^{0,1}([0, T] \times H, \mathbb{R})$ such that the map $(t, x) \mapsto t^\alpha \nabla f(t, x)$ belongs to $C_b([0, T] \times H, \mathbb{R})$. The space $C_\alpha^{0,1}([0, T] \times H)$ is a Banach space when endowed with the norm*

$$\|f\|_{C_\alpha^{0,1}([0, T] \times H)} = \sup_{(t, x) \in [0, T] \times H} |f(t, x)| + \sup_{(t, x) \in (0, T] \times H} t^\alpha \|\nabla f(t, x)\|_H.$$

When clear from the context we will write simply $\|f\|_{C_\alpha^{0,1}}$.

We also denote by $C_\alpha^{0,1,C}([0, T] \times H)$ the space of functions $f \in C_b([0, T] \times H) \cap C^{0,1,C}([0, T] \times H)$ such that the map $(t, x) \mapsto t^\alpha \nabla^C f(t, x)$ belongs to $C_b([0, T] \times H, K)$. The space $C_\alpha^{0,1,C}([0, T] \times H)$ is a Banach space when endowed with the norm

$$\|f\|_{C_\alpha^{0,1,C}([0, T] \times H)} = \sup_{(t, x) \in [0, T] \times H} |f(t, x)| + \sup_{(t, x) \in (0, T] \times H} t^\alpha \|\nabla^C f(t, x)\|_K.$$

When clear from the context we will write simply $\|f\|_{C_\alpha^{0,1,C}}$.

- For any $\alpha \in (0, 1)$ and $T > 0$ we denote by $C_\alpha^{0,2}([0, T] \times H)$ the space of functions $f \in C_b([0, T] \times H) \cap C^{0,2}([0, T] \times H)$ such that for all $t \in (0, T]$, $x \in H$ the map $(t, x) \mapsto t^\alpha \nabla^C \nabla f(t, x)$ is bounded and continuous as a map from $(0, T] \times H$ with values in H . The space $C_\alpha^{0,2}([0, T] \times H)$ turns out to be a Banach space if it is endowed with the norm

$$\begin{aligned} \|f\|_{C_\alpha^{0,2}([0, T] \times H)} &= \sup_{(t, x) \in [0, T] \times H} |f(t, x)| + \sup_{(t, x) \in [0, T] \times H} \|\nabla f(t, x)\|_H + \sup_{(t, x) \in (0, T] \times H} t^\alpha \|\nabla^2 f(t, x)\|_{H \times H}. \end{aligned}$$

We also denote by $C_\alpha^{0,2,C}([0, T] \times H)$ the space of functions $f \in C_b([0, T] \times H) \cap C^{0,2,C}([0, T] \times H)$ such that for all $t \in (0, T]$, $x \in H$ the map $(t, x) \mapsto t^\alpha \nabla^C \nabla f(t, x)$ is bounded and continuous as a map from $(0, T] \times H$ with values in $H \times K$. The space $C_\alpha^{0,2,C}([0, T] \times H)$ turns out to be a Banach space if it is endowed with the norm

$$\begin{aligned} \|f\|_{C_\alpha^{0,2,C}([0, T] \times H)} &= \sup_{(t, x) \in [0, T] \times H} |f(t, x)| + \sup_{(t, x) \in [0, T] \times H} \|\nabla f(t, x)\|_H + \sup_{(t, x) \in (0, T] \times H} t^\alpha \|\nabla^C \nabla f(t, x)\|_{H \times K}. \end{aligned}$$

2.2 Setting of the problem and main assumptions

In this paper we are concerned with the solution of an optimal control problem related to an n -dimensional controlled equation with delay in the control, that we are going to introduce.

In a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider

$$\begin{cases} dy(t) = a_0 y(t) dt + b_0 u(t) dt + \int_{-d}^0 b_1(\xi) u(t + \xi) d\xi + \sigma dW_t, & t \in [0, T] \\ y(0) = y_0, \\ u(\xi) = u_0(\xi), & \xi \in [-d, 0), \end{cases} \quad (2.2)$$

where we assume the following.

Hypothesis 2.3

- (i) W is a standard Brownian motion in \mathbb{R}^k , and $(\mathcal{F}_t)_{t \geq 0}$ is the augmented filtration generated by W ;
- (ii) $a_0 \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$, σ is in $\mathcal{L}(\mathbb{R}^k; \mathbb{R}^n)$;
- (iii) the control strategy u belongs to \mathcal{U} where

$$\mathcal{U} := \{z \in L^2_{\mathcal{P}}(\Omega \times [0, T], \mathbb{R}^m) : u(t) \in U \text{ a.s.}\}$$

where U is a closed subset of \mathbb{R}^n ;

- (iv) $d > 0$ (the maximum delay the control takes to affect the system);
- (v) $b_0 \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$;
- (vi) $b_1 \in L^2([-d, 0], \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n))$.

Notice that assumption (vi) on b_1 does not cover the case of pointwise delay since it is technically complicated to deal with: indeed it gives rise, as we are going to see in next subsection, to an unbounded control operator B , for this reason we leave the extension of our approach to this case for further research.

As noticed in [24], our results can be generalized to the case when the process y is infinite dimensional, more precisely, when y is the solution of the following controlled stochastic differential equation in an infinite dimensional Hilbert space H , with delay in the control:

$$\begin{cases} dy(t) = A_0 y(t)dt + B_0 u(t)dt + \int_{-d}^0 B_1(\xi)u(t+\xi)d\xi + \sigma dW_t, & t \in [0, T] \\ y(0) = y_0, \\ u(\xi) = u_0(\xi), & \xi \in [-d, 0), \end{cases} \quad (2.3)$$

where A_0, B_0, B_1 are suitable operators, see [24] for more details.

2.3 Infinite dimensional reformulation

Following the approach of [31], applied in [22] to the stochastic case, we reformulate equation (2.2) as an abstract stochastic differential equation in the Hilbert space $\mathcal{H} = \mathbb{R}^n \times L^2([-d, 0], \mathbb{R}^n)$. To this end we introduce the operator $A : \mathcal{D}(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ as follows: for $(y_0, y_1) \in \mathcal{H}$

$$A(y_0, y_1) = (a_0 y_0 + y_1(0), -y_1'), \quad \mathcal{D}(A) = \{(y_0, y_1) \in \mathcal{H} : y_1 \in W^{1,2}([-d, 0], \mathbb{R}^n), y_1(-d) = 0\}. \quad (2.4)$$

We denote by e^{tA} the C_0 -semigroup generated by A : for $y = (y_0, y_1) \in \mathcal{H}$,

$$e^{tA} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} e^{ta_0} y_0 + \int_{-d}^0 1_{[-t, 0]} e^{(t+s)a_0} y_1(s) ds \\ y_1(\cdot - t) 1_{[-d+t, 0]}(\cdot) \end{pmatrix} \quad (2.5)$$

We will use, for $N \in \mathbb{N}$ big enough, the resolvent operator $(N - A)^{-1}$ which can be computed explicitly giving

$$(N - A)^{-1} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} (N - a_0)^{-1} \left[y_0 + \int_{-d}^0 e^{Ns} y_1(s) ds \right] \\ \int_{-d}^0 e^{N(s-\cdot)} y_1(s) ds \end{pmatrix} \quad (2.6)$$

Similarly, denoting by $e^{tA^*} = (e^{tA})^*$ the C_0 -semigroup generated by A^* , we have for $z = (z_0, z_1) \in \mathcal{H}$

$$e^{tA^*} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} e^{ta_0^*} z_0 \\ e^{(\cdot+t)a_0^*} z_0 1_{[-t, 0]}(\cdot) + z_1(\cdot + t) 1_{[-d, -t]}(\cdot) \end{pmatrix} \quad (2.7)$$

The infinite dimensional noise operator is defined as

$$G : \mathbb{R}^k \rightarrow \mathcal{H}, \quad Gy = (\sigma y, 0), \quad y \in \mathbb{R}^k. \quad (2.8)$$

The control operator B is bounded and defined as

$$B : \mathbb{R}^m \rightarrow \mathcal{H}, \quad Bu = (b_0 u, b_1(\cdot)u), \quad u \in \mathbb{R}^m \quad (2.9)$$

and its adjoint is

$$B^* : \mathcal{H}^* \rightarrow \mathbb{R}^m, \quad B^*(x_0, x_1) = b_0^* x_0 + \int_{-d}^0 b_1(\xi)^* x_1(\xi) d\xi, \quad (x_0, x_1) \in \mathcal{H}. \quad (2.10)$$

Note that, in the case of pointwise delay the last term of the drift in the state equation (2.2) is $u(t-d)$, hence $b_1(\cdot)$ is a measure: the Dirac delta δ_{-d} . Hence in this case B is unbounded as it takes values in $\mathbb{R}^n \times C^*([-d, 0], \mathbb{R}^n)$ (here we denote by $C^*([-d, 0], \mathbb{R}^n)$ the dual space of $C([-d, 0], \mathbb{R}^n)$).

Given any initial datum $(y_0, u_0) \in \mathcal{H}$ and any admissible control $u \in \mathcal{U}$ we call $y(t; y_0, u_0, u)$ (or simply $y(t)$ when clear from the context) the unique solution (which comes from standard results on SDE's, see e.g. [25] Chapter 4, Sections 2 and 3) of (2.2).

Let us now define the process $Y = (Y_0, Y_1) \in L_{\mathcal{P}}^2(\Omega \times [0, T], \mathcal{H})$ as

$$Y_0(t) = y(t), \quad Y_1(t)(\xi) = \int_{-d}^{\xi} u(\zeta + t - \xi) b_1(\zeta) d\zeta,$$

where y is the solution of equation (2.2) and u is the control process in (2.2). By Proposition 2 of [22], the process Y is the unique solution of the abstract evolution equation in \mathcal{H}

$$\begin{cases} dY(t) = AY(t)dt + Bu(t)dt + GdW_t, & t \in [0, T] \\ Y(0) = y = (y_0, y_1), \end{cases} \quad (2.11)$$

where $y_0 = x_0$ and $y_1(\xi) = \int_{-d}^{\xi} u_0(\zeta - \xi)b_1(\zeta)d\zeta$. Note that we have $y_1 \in L^2([-d, 0]; \mathbb{R}^n)$. Taking the integral (or mild) form of (2.11) we have

$$Y(t) = e^{tA}y + \int_0^t e^{(t-s)A}Bu(s)ds + \int_0^t e^{(t-s)A}GdW_s, \quad t \in [0, T]. \quad (2.12)$$

2.4 Optimal Control problem

The objective is to minimize, over all controls in \mathcal{U} , the following finite horizon cost:

$$J(t, x, u) = \mathbb{E} \int_t^T (\bar{\ell}_0(s, y(s)) + \bar{\ell}_1(u(s))) ds + \mathbb{E}\bar{\phi}(x(T)). \quad (2.13)$$

where $\bar{\ell}_0 : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $\bar{\phi} : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous and bounded while $\bar{\ell}_1 : U \rightarrow \mathbb{R}$ is measurable and bounded from below. Referring to the abstract formulation (2.11) the cost in (2.13) can be rewritten also as

$$J(t, x; u) = \mathbb{E} \left(\int_t^T [\ell_0(s, Y(s)) + \ell_1(u(s))] ds + \phi(Y(T)) \right), \quad (2.14)$$

where $\ell_0 : [0, T] \times \mathcal{H} \rightarrow \mathbb{R}$, $\ell_1 : U \rightarrow \mathbb{R}$ are defined by setting

$$\ell_0(t, x) := \bar{\ell}_0(t, x_0) \quad \forall x = (x_0, x_1) \in \mathcal{H} \quad (2.15)$$

$$\ell_1 := \bar{\ell}_1 \quad (2.16)$$

(here we cut the bar only to keep the notation homogeneous) while $\phi : \mathcal{H} \rightarrow \mathbb{R}$ is defined as

$$\phi(x) := \bar{\phi}(x_0) \quad \forall x = (x_0, x_1) \in \mathcal{H}. \quad (2.17)$$

Clearly, under the assumption above, ℓ_0 and ϕ are continuous and bounded while ℓ_1 is measurable and bounded from below. The value function of the problem is

$$V(t, x) := \inf_{u \in \mathcal{U}} J(t, x; u). \quad (2.18)$$

As done in Subsection 1.1, we define the Hamiltonian in a modified way (see (1.7)); indeed, for $p \in \mathcal{H}$, $u \in U$, we define the current value Hamiltonian H_{CV} as

$$H_{CV}(p; u) := \langle p, u \rangle_{\mathbb{R}^m} + \ell_1(u)$$

and the (minimum value) Hamiltonian by

$$H_{min}(p) = \inf_{u \in U} H_{CV}(p; u), \quad (2.19)$$

The associated HJB equation with unknown v is then formally written as

$$\begin{cases} -\frac{\partial v(t, x)}{\partial t} = \frac{1}{2} Tr GG^* \nabla^2 v(t, x) + \langle Ax, \nabla v(t, x) \rangle_{\mathcal{H}} + \ell_0(t, x) + H_{min}(\nabla^B v(t, x)), & t \in [0, T], x \in D(A), \\ v(T, x) = \phi(x). \end{cases} \quad (2.20)$$

Existence of mild solutions of (2.20) is proved in [24], and the following assumptions are needed.

Hypothesis 2.4

(i) $\phi \in C_b(\mathcal{H})$ and it is given by (2.17) for a suitable $\phi \in C_b(\mathbb{R}^n)$;

- (ii) $\ell_0 \in C_b([0, T] \times \mathcal{H})$ and it is given by (2.15) for a suitable $\bar{\ell}_0 \in C_b([0, T] \times \mathbb{R}^n)$;
- (iii) $\ell_1 : U \rightarrow \mathbb{R}$ is measurable and bounded from below;
- (iv) the Hamiltonian $H_{\min} : \mathbb{R}^m \rightarrow \mathbb{R}$ is Lipschitz continuous so there exists $L > 0$ such that

$$\begin{aligned} |H_{\min}(p_1) - H_{\min}(p_2)| &\leq L|p_1 - p_2| \quad \forall p_1, p_2 \in \mathbb{R}^m; \\ |H_{\min}(p)| &\leq L(1 + |p|) \quad \forall p \in \mathbb{R}^m. \end{aligned} \quad (2.21)$$

To get more regular solutions (well defined second derivative $\nabla^B \nabla$, which will be used to prove existence of optimal feedback controls) we will need the following further assumption.

Hypothesis 2.5

- (i) ℓ_0 is continuously differentiable in the variable x with bounded derivative.
- (ii) the Hamiltonian $H_{\min} : \mathbb{R}^m \rightarrow \mathbb{R}$ is continuously differentiable and, for a given $L > 0$, we have, beyond (2.21),

$$|\nabla H_{\min}(p_1) - \nabla H_{\min}(p_2)| \leq L|p_1 - p_2| \quad \forall p_1, p_2 \in \mathbb{R}^m; \quad (2.22)$$

3 The Ornstein-Uhlenbeck semigroup and the HJB equation

In the setting of Section 2.2 we assume that Hypothesis 2.3 holds true. We take $\mathcal{H} = \mathbb{R}^n \times L^2(-d, 0; \mathbb{R}^n)$, $\Xi = \mathbb{R}^k$, $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space, W a standard Wiener process in Ξ , A and G as in (2.4) and (2.8). Then, for $x \in \mathcal{H}$, we take the Ornstein-Uhlenbeck process $X^x(\cdot)$ given by

$$\begin{cases} dX(t) = AX(t)dt + GdW_t, & t \geq 0 \\ X(0) = x, \end{cases} \quad (3.1)$$

In mild form, the Ornstein-Uhlenbeck process X^x is given by

$$X^x(t) = e^{tA}x + \int_0^t e^{(t-s)A}GdW_s, \quad t \geq 0. \quad (3.2)$$

X is a Gaussian process, namely for every $t > 0$, the law of $X(t)$ is $\mathcal{N}(e^{tA}x, Q_t)$, the Gaussian measure with mean $e^{tA}x$ and covariance operator Q_t , where

$$Q_t = \int_0^t e^{sA}GG^*e^{sA^*}ds.$$

The associated Ornstein-Uhlenbeck transition semigroup R_t is defined by setting, for all $f \in B_b(\mathcal{H})$,

$$R_t[f](x) = \mathbb{E}f(X^x(t)) = \int_K f(z + e^{tA}x)\mathcal{N}(0, Q_t)(dz). \quad (3.3)$$

Given any $\bar{\phi} \in B_b(\mathbb{R}^n)$, we define, as in (2.17) a function $\phi \in B_b(\mathcal{H})$, by setting

$$\phi(x) = \bar{\phi}(x_0) \quad \forall x = (x_0, x_1) \in \mathcal{H}. \quad (3.4)$$

For such functions, the Ornstein-Uhlenbeck semigroup R_t is written as

$$R_t[\phi](x) = \mathbb{E}\phi(X^x(t)) = \mathbb{E}\bar{\phi}((X^x(t))_0) = \int_{\mathcal{H}} \bar{\phi}((z + e^{tA}x)_0)\mathcal{N}(0, Q_t)(dz). \quad (3.5)$$

Concerning the covariance operator Q_t , by Lemma 4.6 in [24] we have that

$$\text{Im } Q_t = \text{Im } Q_t^0 \times \{0\} \subseteq \mathbb{R}^n \times \{0\}$$

where Q_t^0 is the selfadjoint operator in \mathbb{R}^n defined as

$$Q_t^0 := \int_0^t e^{sa_0}\sigma\sigma^*e^{sa_0^*}ds. \quad (3.6)$$

Then for every $(x_0, x_1) \in \mathcal{H}$ we have

$$Q_t(x_0, x_1) = (Q_t^0 x_0, 0) \quad (3.7)$$

and so for every $\phi : \mathcal{H} \rightarrow \mathbb{R}$ defined in (3.4) we have

$$R_t[\phi](x) = \int_{\mathbb{R}^n} \bar{\phi}(z_0 + (e^{tA}x)_0) \mathcal{N}(0, Q_t^0)(dz_0). \quad (3.8)$$

For the Ornstein Uhlenbeck transition semigroup we have the following regularizing property.

Proposition 3.1 *Assume that Hypothesis 2.3 holds. Assume moreover that, either*

$$\text{Im}(e^{ta_0}b_0) \subseteq \text{Im } \sigma, \quad \forall t > 0; \quad \text{Im } b_1(s) \in \text{Im } \sigma, \quad a.e. \forall s \in [-d, 0] \quad (3.9)$$

or

$$\text{Im} \left(e^{ta_0}b_0 + \int_{-d}^0 1_{[-t, 0]} e^{(t+r)a_0} b_1(dr) \right) \subseteq \text{Im } \sigma, \quad \forall t > 0. \quad (3.10)$$

Then, for any bounded measurable ϕ as in (3.4), $R_t[\phi]$ is B -Fréchet differentiable for every $t > 0$, and, for every $h \in \mathbb{R}^m$, $\langle \nabla^B(R_t[\phi])(x), k \rangle_{\mathbb{R}^m}$ is given by

$$\langle \nabla^B(R_t[\phi])(x), k \rangle_{\mathbb{R}^m} = \int_{\mathbb{R}^n} \bar{\phi}(z_0 + (e^{tA}x)_0) \left\langle (Q_t^0)^{-1/2} (e^{tA}Bk)_0, (Q_t^0)^{-1/2} z_0 \right\rangle_{\mathbb{R}^n} \mathcal{N}(0, Q_t^0)(dz_0). \quad (3.11)$$

Moreover, for every $k \in \mathbb{R}^m$,

$$|\langle \nabla^B(R_t[\phi])(x), k \rangle_{\mathbb{R}^m}| \leq \|\bar{\phi}\|_\infty \left\| (Q_t^0)^{-1/2} (e^{tA}B)_0 \right\|_{\mathcal{L}(R^m, \mathbb{R}^n)} |k|_{\mathbb{R}^m}, \quad (3.12)$$

so that for all $T > 0$ there exists C_T such that

$$|\langle \nabla^B(R_t[\phi])(x), k \rangle_{\mathbb{R}^m}| \leq C_T t^{-1/2} \|\bar{\phi}\|_\infty |k|_{\mathbb{R}^m}. \quad (3.13)$$

3.1 Regular mild solutions of the HJB equation

We now recall results proved in [24] about existence of mild solutions of the HJB equation (1.5). We also state results about the existence of second order derivatives that will be need in Section 5.2 to solve our control problem.

First of all we introduce some suitable spaces of differentiable functions in $C_b([0, T] \times \mathcal{H})$. We fix $\alpha \in (0, 1)$, in the following we will consider these spaces with $\alpha = 1/2$.

Definition 3.2 *Let $T > 0$, $\alpha \in (0, 1)$. A function $g \in C_b([0, T] \times \mathcal{H})$ belongs to $\Sigma_{T, \alpha}^1$ if there exists a function $f \in C_{\alpha}^{0,1}([0, T] \times \mathbb{R}^n)$ such that*

$$g(t, x) = f(t, (e^{tA}x)_0), \quad \forall (t, x) \in [0, T] \times \mathcal{H}.$$

If $g \in \Sigma_{T, \alpha}^1$, for any $t \in (0, T]$ the function $g(t, \cdot)$ is both Fréchet differentiable and B -Fréchet differentiable. Moreover, for $(t, x) \in [0, T] \times \mathcal{H}$, $h \in \mathcal{H}$, $k \in \mathbb{R}^m$,

$$\langle \nabla g(t, x), h \rangle_{\mathcal{H}} = \langle \nabla f(t, (e^{tA}x)_0), (e^{tA}h)_0 \rangle_{\mathbb{R}^n}, \quad \text{and} \quad \langle \nabla^B g(t, x), k \rangle_{\mathbb{R}^m} = \langle \nabla f(t, (e^{tA}x)_0), (e^{tA}Bk)_0 \rangle_{\mathbb{R}^n}.$$

This in particular imply that, for all $k \in \mathbb{R}^m$

$$\langle \nabla^B g(t, x), k \rangle_{\mathbb{R}^m} = \langle \nabla g(t, x), Bk \rangle_{\mathbb{R}^n \times L^2([-d, 0]; \mathbb{R}^n)}, \quad (3.14)$$

which also means $B^* \nabla g = \nabla^B g$. For later notational use we call $\bar{f} \in C_b((0, T] \times \mathbb{R}^n; \mathbb{R}^m)$ the function defined by

$$\langle \bar{f}(t, y), k \rangle_{\mathbb{R}^m} = t^\alpha \langle \nabla f(t, y), (e^{tA}Bk)_0 \rangle_{\mathbb{R}^n}, \quad (t, y) \in (0, T] \times \mathbb{R}^n, \quad k \in \mathbb{R}^m,$$

which is such that

$$t^\alpha \nabla^B g(t, x) = \bar{f}(t, (e^{tA}x)_0).$$

We also notice that if $g \in \Sigma_{T, \alpha}^1$, then in order to have g B -Fréchet differentiable it suffices to require $(e^{tA}B)_0$ bounded and continuous. The set $\Sigma_{T, \alpha}^1$ is a closed subspace of $C_{\alpha}^{0,1,B}([0, T] \times \mathcal{H})$.

Next, in analogy to what we have done defining $\Sigma_{T, \alpha}^1$, we introduce a subspace $\Sigma_{T, \alpha}^2$ of functions $g \in C_{\alpha}^{0,2,B}([0, T] \times \mathcal{H})$ that depends in a special way on the variable $x \in \mathcal{H}$.

Definition 3.3 A function $g \in C_b([0, T] \times \mathcal{H})$ belongs to $\Sigma_{T, \alpha}^2$ if there exists a function $f \in C_{\alpha}^{0,2}([0, T] \times \mathbb{R}^n)$ such that for all $(t, x) \in [0, T] \times \mathcal{H}$,

$$g(t, x) = f(t, (e^{tA}x)_0).$$

If $g \in \Sigma_{T, \alpha}^2$ then for any $t \in (0, T]$ the function $g(t, \cdot)$ is Fréchet differentiable and

$$\langle \nabla g(t, x), h \rangle_{\mathcal{H}} = \langle \nabla f(t, (e^{tA}x)_0), (e^{tA}h)_0 \rangle_{\mathbb{R}^n}, \quad \text{for } (t, x) \in [0, T] \times \mathcal{H}, h \in \mathcal{H}.$$

Moreover also $\nabla g(t, \cdot)$ is B -Fréchet differentiable and

$$\langle \nabla^B(\nabla g(t, x)h), k \rangle_{\mathbb{R}^m} = \langle \nabla^2 f(t, (e^{tA}x)_0)(e^{tA}h)_0, (e^{tA}Bk)_0 \rangle_{\mathbb{R}^n}, \quad \text{for } (t, x) \in [0, T] \times \mathcal{H}, h \in \mathcal{H}, k \in \mathbb{R}^m.$$

We also notice that, since the function f is twice continuously Fréchet differentiable the second order derivatives $\nabla^B \nabla g$ and $\nabla \nabla^B g$ both exist and coincide:

$$\langle \nabla^B \langle \nabla g(t, x), h \rangle_{\mathcal{H}}, k \rangle_{\mathbb{R}^m} = \langle \nabla \langle \nabla^B g(t, x), k \rangle_{\mathbb{R}^m}, h \rangle_{\mathcal{H}}.$$

Again for later notational use we call $\bar{f}_1 \in C_b([0, T] \times \mathbb{R}^n; \mathbb{R}^m)$ the function defined by

$$\langle \bar{f}_1(t, y), h \rangle_{\mathbb{R}^m} = \langle \nabla f(t, y), (e^{tA}Bh)_0 \rangle_{\mathbb{R}^n}, \quad (t, y) \in [0, T] \times \mathbb{R}^n, \quad h \in \mathbb{R}^m,$$

which is such that

$$\nabla^B g(t, x) = \bar{f}_1(t, (e^{tA}x)_0).$$

Similarly we call $\bar{\bar{f}} \in C_b((0, T] \times \mathbb{R}^n; \mathcal{L}(\mathcal{H}, \mathbb{R}^m))$ the function defined by

$$\left\langle \left\langle \bar{\bar{f}}(t, y), h \right\rangle_{\mathcal{H}}, k \right\rangle_{\mathbb{R}^m} = t^{\alpha} \langle \nabla^2 f(t, y)(e^{tA}h)_0, (e^{tA}Bk)_0 \rangle_{\mathbb{R}^n} \quad (t, y) \in [0, T] \times \mathbb{R}^n, \quad h \in \mathcal{H}, k \in \mathbb{R}^m,$$

which is such that

$$t^{\alpha} \nabla^B \nabla g(t, x) = t^{\alpha} \nabla \nabla^B g(t, x) = \bar{\bar{f}}(t, (e^{tA}x)_0).$$

When (3.9) or (3.10) hold we know, by Proposition 3.1, that the function $g(t, x) = R_t[\phi](x)$ for ϕ given by (3.4) with $\bar{\phi}$ bounded and continuous, belongs to $\Sigma_{T, 1/2}^1$.

Lemma 3.4 Let (3.9) or (3.10) hold true. Let $T > 0$ and let $\psi : \mathbb{R}^m \rightarrow \mathbb{R}$ be a continuous function satisfying Hypothesis (2.4), estimates (2.21). Then

i) for every $g \in \Sigma_{T, 1/2}^1$, the function $\hat{g} : [0, T] \times \mathcal{H} \rightarrow \mathbb{R}$ belongs to $\Sigma_{T, 1/2}^1$ where

$$\hat{g}(t, x) = \int_0^t R_{t-s}[\psi(\nabla^B g(s, \cdot))](x) ds. \quad (3.15)$$

Hence, in particular, $\hat{g}(t, \cdot)$ is B -Fréchet differentiable for every $t \in (0, T]$ and, for all $x \in \mathcal{H}$,

$$|\nabla^B(\hat{g}(t, \cdot))(x)|_{(\mathbb{R}^m)^*} \leq C \left(t^{1/2} + \|g\|_{C_{1/2}^{0,1}} \right). \quad (3.16)$$

If σ is onto, then $\hat{g}(t, \cdot)$ is Fréchet differentiable for every $t \in (0, T]$ and, for all $h \in \mathcal{H}$, $x \in \mathcal{H}$,

$$|\nabla(\hat{g}(t, \cdot))(x)|_{\mathcal{H}^*} \leq C \left(t^{1/2} + \|g\|_{C_{1/2}^{0,1}} \right). \quad (3.17)$$

ii) Assume moreover that $\psi \in C^1(\mathbb{R}^m)$. For every $g \in \Sigma_{T, 1/2}^2$, the function \hat{g} defined in (3.15) belongs to $\Sigma_{T, 1/2}^2$. Hence, in particular, the second order derivatives $\nabla \nabla^B \hat{g}(t, \cdot)$ and $\nabla^B \nabla \hat{g}(t, \cdot)$ exist, coincide and for every $t \in (0, T]$ and, for all $x \in \mathcal{H}$,

$$|\nabla^B \nabla(\hat{g}(t, \cdot))(x)|_{\mathcal{H}^* \times (\mathbb{R}^m)^*} \leq C \|g\|_{C_{1/2}^{0,2,B}} \quad (3.18)$$

If σ is onto, then $\hat{g}(t, \cdot)$ is twice Fréchet differentiable and for every $t \in (0, T]$, for all $h \in \mathcal{H}$ and $x \in \mathcal{H}$,

$$|\nabla^2(\hat{g}(t, \cdot))(x)|_{\mathcal{H}^* \times \mathcal{H}^*} \leq C \|g\|_{C_{1/2}^{0,2}} \quad (3.19)$$

Now we introduce mild solutions to the HJB equation (2.20). By applying formally the variation of constants formula to (2.20) we get the integral equation satisfied by the mild solution, that we rewrite here for the reader convenience:

$$v(t, x) = R_{T-t}[\phi](x) + \int_t^T R_{s-t} [H_{min}(\nabla^B v(s, \cdot)) + \ell_0(s, \cdot)](x) ds, \quad t \in [0, T], x \in H. \quad (3.20)$$

We use this formula to give the notion of mild solution for the HJB equation (2.20).

Definition 3.5 *We say that a function $v : [0, T] \times \mathcal{H} \rightarrow \mathbb{R}$ is a mild solution of the HJB equation (2.20) if the following are satisfied:*

1. $v(T - \cdot, \cdot) \in C_{1/2}^{0,1,B}([0, T] \times \mathcal{H})$;
2. equality (3.20) holds on $[0, T] \times \mathcal{H}$.

Theorem 3.6 *Let Hypotheses 2.3 and 2.4 hold and let (3.9) or (3.10) hold. Then the HJB equation (2.20) admits a mild solution v according to Definition 3.5. Moreover v is unique among the functions w such that $w(T - \cdot, \cdot) \in \Sigma_{T,1/2}$ and it satisfies, for suitable $C_T > 0$, the estimate*

$$\|v(T - \cdot, \cdot)\|_{C_{1/2}^{0,1,B}} \leq C_T (\|\bar{\phi}\|_\infty + \|\bar{\ell}_0\|_\infty). \quad (3.21)$$

Finally if the initial datum ϕ is also continuously B -Fréchet (or Fréchet) differentiable, then $v \in C_b^{0,1,B}([0, T] \times \mathcal{H})$ and, for suitable $C_T > 0$,

$$\|v\|_{C_b^{0,1,B}} \leq C_T (\|\phi\|_\infty + \|\nabla^B \phi\|_\infty + \|\ell_0\|_\infty) \quad (3.22)$$

(substituting $\nabla^B \phi$ with $\nabla \phi$ if ϕ is Fréchet differentiable). If σ is onto, then the mild solution of equation (2.20) found in the previous theorem is also Fréchet differentiable, and the following estimate holds true

$$\|v(T - \cdot, \cdot)\|_{C_{1/2}^{0,1}} \leq C_T (\|\bar{\phi}\|_\infty + \|\bar{\ell}_0\|_\infty) \quad (3.23)$$

If the coefficients are regular, we have further regularity on the mild solution, namely:

- (i) if ϕ is continuously differentiable then we have $v \in \Sigma_{T,1/2}^2$, hence the second order derivatives $\nabla^B \nabla v$ and $\nabla \nabla^B v$ exist and are equal. Moreover there exists a constant $C > 0$ such that

$$|\nabla v(t, x)| \leq C (\|\nabla \bar{\phi}\|_\infty + \|\nabla \bar{\ell}_0\|_\infty), \quad (3.24)$$

$$|\nabla^B \nabla v(t, x)| = |\nabla \nabla^B v(t, x)| \leq C \left((T-t)^{-1/2} \|\nabla \bar{\phi}\|_\infty + (T-t)^{1/2} \|\nabla \bar{\ell}_0\|_\infty \right). \quad (3.25)$$

Finally, if σ is onto, then also $\nabla^2 v$ exists and is continuous and, for suitable $C > 0$,

$$|\nabla^2 v(t, x)| \leq C \left((T-t)^{-1/2} \|\nabla \bar{\phi}\|_\infty + (T-t)^{1/2} \|\nabla \bar{\ell}_0\|_\infty \right). \quad (3.26)$$

- (ii) If ϕ is only continuous then the function $(t, x) \mapsto (T-t)^{1/2} v(t, x)$ belongs to $\Sigma_{T,1/2}^2$. Moreover there exists a constant $C > 0$ such that

$$|\nabla v(t, x)| \leq C \left((T-t)^{-1/2} \|\bar{\phi}\|_\infty + \|\nabla \bar{\ell}_0\|_\infty \right), \quad (3.27)$$

$$|\nabla^B \nabla v(t, x)| = |\nabla \nabla^B v(t, x)| \leq C \left((T-t)^{-1} \|\bar{\phi}\|_\infty + (T-t)^{-1/2} \|\nabla \bar{\ell}_0\|_\infty \right). \quad (3.28)$$

Finally, if σ is onto, then also $\nabla^2 v$ exists and is continuous in $[0, T) \times \mathcal{H}$ and, for suitable $C > 0$,

$$|\nabla^2 v(t, x)| \leq C \left((T-t)^{-1} \|\nabla \bar{\phi}\|_\infty + (T-t)^{-1/2} \|\nabla \bar{\ell}_0\|_\infty \right). \quad (3.29)$$

4 \mathcal{K} -convergence and approximations of solutions of the HJB equation

We first introduce the notion of \mathcal{K} -convergence, following [6] and [20].

Definition 4.1 A sequence $(f_n)_{n \geq 0} \in C_b(\mathcal{H})$ is said to be \mathcal{K} -convergent to a function $f \in C_b(\mathcal{H})$ (and we shall write $f_n \xrightarrow{\mathcal{K}} f$ or $f = \mathcal{K} - \lim_{n \rightarrow \infty} f_n$) if for any compact set $K \subset \mathcal{H}$

$$\sup_{n \in \mathbb{N}} \|f_n\|_\infty < +\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{x \in K} |f(x) - f_n(x)| = 0.$$

Similarly, given $I \subseteq \mathbb{R}$, a sequence $(f_n)_{n \geq 0} \in C_b(I \times \mathcal{H})$ is said to be \mathcal{K} -convergent to a function $f \in C_b(I \times \mathcal{H})$ (and we shall write again $f_n \xrightarrow{\mathcal{K}} f$ or $f = \mathcal{K} - \lim_{n \rightarrow \infty} f_n$) if for any compact set $K \in \mathcal{H}$ and for any $(t, x) \in I \times K$ we have

$$\sup_{n \in \mathbb{N}} \|f_n\|_\infty < +\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{(t, x) \in I \times K} |f(t, x) - f_n(t, x)| = 0.$$

Now we recall the definition (given in [10], beginning of Chapter 7) of *strict solution* of a family of Kolmogorov equations. Consider the following forward Kolmogorov equation with unknown w :

$$\begin{cases} \frac{\partial w(t, x)}{\partial t} = \frac{1}{2} \text{Tr} \, GG^* \nabla^2 w(t, x) + \langle Ax, \nabla w(t, x) \rangle + \mathcal{F}(t, x) & t \in [0, T], x \in \mathcal{H}, \\ w(0, x) = \phi(x). \end{cases} \quad (4.1)$$

where the functions $\mathcal{F} : [0, T] \times \mathcal{H} \rightarrow \mathbb{R}$ and $\phi : \mathcal{H} \rightarrow \mathbb{R}$ are bounded and continuous.

Definition 4.2 By *strict solution* of the Kolmogorov equation (4.1) we mean a function w such that

$$\begin{cases} w \in C_b([0, T] \times \mathcal{H}) \quad \text{and} \quad w(0, x) = \phi(x) \\ w(t, \cdot) \in UC_b^2(\mathcal{H}), \quad \forall t \in [0, T]; \\ w(\cdot, x) \in C^1([0, T]), \quad \forall x \in D(A) \text{ and } w \text{ satisfies (4.1)}. \end{cases} \quad (4.2)$$

Now we prove a key approximation lemma.

Lemma 4.3 Let Hypothesis 2.3 and 2.4 hold. Let also (3.9) or (3.10) hold. Let v be the mild solution of the HJB equation (2.20) and set $w(t, x) = v(T - t, x)$ for $(t, x) \in [0, T] \times \mathcal{H}$. Then there exist three sequences of functions $(\bar{\phi}_n)$, $(\bar{\ell}_{0,n})$ and (\mathcal{F}_n) such that, for all $n \in \mathbb{N}$,

$$\bar{\phi}_n \in C_c^\infty(\mathbb{R}^n), \quad \bar{\ell}_{0,n} \in C_c^\infty([0, T] \times \mathbb{R}^n), \quad \mathcal{F}_n \in C_c^\infty([0, T] \times \mathcal{H}) \cap \Sigma_{T, 1/2} \quad (4.3)$$

and

$$\bar{\phi}_n \rightarrow \bar{\phi}, \quad \bar{\ell}_{0,n} \rightarrow \bar{\ell}, \quad \mathcal{F}_n \rightarrow H_{\min}(\nabla^B w) \quad (4.4)$$

in the sense of \mathcal{K} -convergence. Moreover, defining $\phi_n(x) = \bar{\phi}_n(x_0)$, $\ell_{0,n}(s, x) = \bar{\ell}_{0,n}(s, x_0)$ and

$$w_n(t, x) := R_t \phi_n + \int_0^t R_{t-s} [\mathcal{F}_n(s, \cdot) + \ell_{0,n}(s, \cdot)](x) ds \quad (4.5)$$

the following hold:

- $w_n \in UC_b^{1,2}([0, T] \times \mathcal{H}) \cap \Sigma_{T, 1/2}$,
- w_n is a strict solution of (4.1) with ϕ_n in place of ϕ and $\mathcal{F}_n + \ell_{0,n}$ in place of \mathcal{F} ,
- we have, in the sense of \mathcal{K} -convergence (the first in $[0, T] \times \mathcal{H}$, the second in $(0, T] \times \mathcal{H}$),

$$w_n \rightarrow w, \quad t^{1/2} \nabla^B w_n \rightarrow t^{1/2} \nabla^B w. \quad (4.6)$$

Proof. We divide the proof in three steps.

Step 1: choosing the three approximating sequences. We choose $\bar{\phi}_n$ and $\bar{\ell}_{0,n}$ to be the standard approximations by convolution of $\bar{\phi}$ and $\bar{\ell}_0$. To define \mathcal{F}_n we observe first that, since $w \in \Sigma_{T,1/2}^1$, then the function

$$(t, x) \mapsto \mathcal{F}(t, x) := H_{min}(\nabla^B w(t, x))$$

has the property that there exist $f : [0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}$, continuous and bounded, such that

$$\mathcal{F}(t, x) = t^{-1/2} f(t, (e^{tA} x)_0).$$

We then let f_n be the approximation by convolution of f and define

$$\mathcal{F}_n(t, x) = t^{-1/2} f_n(t, (e^{tA} x)_0).$$

Step 2: proof that $w_n \in UC_b^{1,2}([0, T] \times \mathcal{H}) \cap \Sigma_{T,1/2}^1$ and that it is a strict solution. The fact that $w_n \in \Sigma_{T,1/2}$ follows immediately from (4.5), Proposition 3.1 and Lemma 3.4-(i). Differentiability with respect to the variable x follows by applying the dominated convergence theorem, while explicitly differentiating $R_t[\phi]$, namely for any $h \in \mathcal{H}$

$$\begin{aligned} \langle \nabla R_t[\phi](x), h \rangle_H &= \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \left[\int_{\mathcal{H}} \phi(z + e^{tA}(x + \alpha h)) \mathcal{N}(0, Q_t)(dz) - \int_{\mathcal{H}} \phi(z + e^{tA}x) \mathcal{N}(0, Q_t)(dz) \right] \\ &= \int_{\mathcal{H}} \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} [\phi(z + e^{tA}(x + \alpha h)) - \phi(z + e^{tA}x)] \mathcal{N}(0, Q_t)(dz) \\ &= \int_{\mathcal{H}} \langle \nabla \phi(z + e^{tA}x), e^{tA}h \rangle_H \mathcal{N}(0, Q_t)(dz) = R_t[\langle \nabla \phi, e^{tA}h \rangle_H](x). \end{aligned}$$

In a similar way, differentiating twice we get that, for all $h, k \in \mathcal{H}$

$$\langle \nabla^2 R_t[\phi_n](x)h, k \rangle_{\mathcal{H}} = R_t[\langle \nabla^2 \phi_n e^{tA}h, e^{tA}k \rangle_{\mathcal{H}}](x).$$

Similarly we have, for the convolution term containing \mathcal{F}_n ,

$$\begin{aligned} &\left\langle \nabla \int_0^t R_{t-s}[\mathcal{F}_n(s, \cdot)](x), h \right\rangle_{\mathcal{H}} \\ &= \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \int_0^t \left[\int_{\mathcal{H}} \mathcal{F}_n(s, z + e^{(t-s)A}(x + \alpha h)) \mathcal{N}(0, Q_{t-s})(dz) - \int_{\mathcal{H}} \mathcal{F}_n(s, z + e^{(t-s)A}x) \mathcal{N}(0, Q_{t-s})(dz) ds \right] \\ &= \int_0^t \int_{\mathcal{H}} \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} [\mathcal{F}_n(s, z + e^{(t-s)A}(x + \alpha h)) - \mathcal{F}_n(s, z + e^{(t-s)A}x)] \mathcal{N}(0, Q_{t-s})(dz) ds \\ &= \int_0^t \int_{\mathcal{H}} \langle \nabla \mathcal{F}_n(s, z + e^{(t-s)A}x), e^{(t-s)A}h \rangle_{\mathcal{H}} \mathcal{N}(0, Q_{t-s})(dz) ds \\ &= \int_0^t R_{t-s}[\langle \nabla \mathcal{F}_n(s, \cdot), e^{(t-s)A}h \rangle_{\mathcal{H}}](x) ds, \end{aligned}$$

and also, arguing in the same way,

$$\left\langle \nabla^2 \int_0^t R_{t-s}[\mathcal{F}_n(s, \cdot)](x) ds h, k \right\rangle_{\mathcal{H}} = \int_0^t R_{t-s}[\langle \nabla^2 \mathcal{F}_n(s, \cdot) e^{(t-s)A}h, e^{(t-s)A}k \rangle_{\mathcal{H}}](x) ds.$$

The convolution term involving $\ell_{0,n}$ is treated exactly in the same way. The proof that w_n is differentiable with respect to time and that $w_{nt} \in UC_b([0, T] \times \mathcal{H})$ is completely analogous to what is done in [9, Theorems 9.23 and 9.25]) for homogeneous Kolmogorov equations and we omit it⁴. By Theorem 5.3 in [7], see also Theorem 7.5.1 in [10] for Kolmogorov equations, we finally conclude that w_n is a strict solution to equation 4.1.

⁴The proof in the nonhomogeneous case can be found in the forthcoming book [12, Section 4.4].

Step 3: proof of the convergences. First we prove that the sequences (w_n) and $(t^{1/2}\nabla^B w_n)$ are bounded uniformly with respect to n . Indeed, by (4.5) and by the properties of convolutions,

$$\begin{aligned} |w_n(t, x)| &\leq \|\bar{\phi}_n\|_\infty + \int_0^t \sup_{x \in \mathcal{H}} [|\mathcal{F}_n(s, x)| + |\ell_{0,n}(s, x)|] ds \\ &\leq \|\bar{\phi}\|_\infty + \int_0^t \sup_{y \in \mathbb{R}^n} [s^{-1/2}|f_n(s, y)| + |\bar{\ell}_{0,n}(s, y)|] ds \leq \|\bar{\phi}\|_\infty + \int_0^t \sup_{y \in \mathbb{R}^n} [s^{-1/2}|f(s, y)| + |\bar{\ell}_0(s, y)|] ds \end{aligned}$$

Moreover, using Proposition 3.1 and the results in [24], Section 5, formula 5.9, with $\psi = \text{identity}$, namely

$$\left\langle \nabla^B \left(\int_0^t R_{t-s} [\nabla^B(g(s, \cdot))] ds \right) (x), k \right\rangle_{\mathbb{R}^m} = \quad (4.7)$$

$$= \int_0^t \int_{\mathcal{H}} s^{-1/2} \bar{f}(s, (e^{sA}z)_0 + (e^{tA}x)_0) \left\langle (Q_{t-s}^0)^{-1/2} (e^{tA}Bk)_0, (Q_{t-s}^0)^{-1/2} z_0 \right\rangle_{\mathbb{R}^n} \mathcal{N}(0, Q_{t-s})(dz) ds,$$

we get

$$|t^{1/2}\nabla^B w_n(t, x)| \leq C\|\bar{\phi}\|_\infty + Ct^{1/2} \int_0^t s^{-1/2}(t-s)^{-1/2} \sup_{y \in \mathbb{R}^n} (|f(s, y)| + |\bar{\ell}_0(s, y)|) ds,$$

for a suitable $C > 0$. Now, with similar computations, we prove the convergences. Indeed,

$$\begin{aligned} |v_n(t, x) - v(t, x)| &\leq \int_{\mathcal{H}} |\bar{\phi}_n((e^{tA}z)_0 + (e^{tA}x)_0) - \bar{\phi}((e^{tA}z)_0 + (e^{tA}x)_0)| \mathcal{N}(0, Q_t)(dz) \\ &\quad + \int_0^t \int_{\mathcal{H}} \left[|s^{-1/2} \mathcal{F}_n(s, (e^{sA}z)_0 + (e^{tA}x)_0) - s^{-1/2} \mathcal{F}(s, (e^{sA}z)_0 + (e^{tA}x)_0)| \right. \\ &\quad \left. + |\ell_{0,n}(s, (e^{sA}z)_0 + (e^{tA}x)_0) - \ell_0(s, (e^{sA}z)_0 + (e^{tA}x)_0)| \right] \mathcal{N}(0, Q_{t-s})(dz) ds. \end{aligned}$$

Since, for every compact set $\mathcal{K} \subset \mathcal{H}$ the set $\{(e^{tA}x)_0, t \in [0, T], x \in \mathcal{K}\}$ is compact in \mathbb{R}^n , then by the Dominated Convergence Theorem we get that for any compact set $\mathcal{K} \subset \mathcal{H}$

$$\sup_{(t,x) \in [0,T] \times \mathcal{K}} |v_n(t, x) - v(t, x)| \rightarrow 0. \quad (4.8)$$

Moreover, using again Propositions 3.1 and 3.4-(i), for a suitable $C > 0$, we get

$$\begin{aligned} |\nabla^B v_n(t, x) - \nabla^B v(t, x)| &\leq C \int_{\mathcal{H}} (\bar{\phi}_n((e^{tA}z)_0 + (e^{tA}x)_0) - \bar{\phi}((e^{tA}z)_0 + (e^{tA}x)_0)) \left\langle (Q_t^0)^{-1/2} (e^{tA}Bh)_0, (Q_t^0)^{-1/2} z_0 \right\rangle_{\mathbb{R}^n} \mathcal{N}(0, Q_t)(dz) \\ &\quad + C \int_0^t \int_{\mathcal{H}} \left(\left| s^{-1/2} f_n(s, (e^{sA}z)_0 + (e^{tA}x)_0) - s^{-1/2} f(s, (e^{sA}z)_0 + (e^{tA}x)_0) \right| \right. \\ &\quad \left. + |\bar{\ell}_{0,n}(s, (e^{sA}z)_0 + (e^{tA}x)_0) - \bar{\ell}_0(s, (e^{sA}z)_0 + (e^{tA}x)_0)| \right) \\ &\quad \left| \left\langle (Q_{t-s}^0)^{-1/2} (e^{(t-s)A}Bh)_0, (Q_{t-s}^0)^{-1/2} z_0 \right\rangle_{\mathbb{R}^n} \right| \mathcal{N}(0, Q_{t-s})(dz) ds. \end{aligned}$$

By Proposition 3.1 we know that for suitable $C > 0$,

$$\int_{\mathcal{H}} \left| \left\langle (Q_t^0)^{-1/2} (e^{tA}Bh)_0, (Q_t^0)^{-1/2} z_0 \right\rangle_{\mathbb{R}^n} \right|^2 \mathcal{N}(0, Q_t)(dz) \leq C \|(Q_t^0)^{-1/2} (e^{tA}Bh)_0\|^2 \leq \frac{C}{t}.$$

Hence, applying Cauchy-Schwartz inequality we get

$$\begin{aligned} |\nabla^B v_n(t, x) - \nabla^B v(t, x)| &\leq Ct^{-1/2} \left| \int_{\mathcal{H}} |\bar{\phi}_n((e^{tA}z)_0 + (e^{tA}x)_0) - \bar{\phi}_n((e^{tA}z)_0 + (e^{tA}x)_0)|^2 \mathcal{N}(0, Q_t)(dz) \right|^{1/2} \\ &\quad + \int_0^t (t-s)^{-1/2} \left(\int_{\mathcal{H}} \left(\left| s^{-1/2} f_n(s, (e^{sA}z)_0 + (e^{tA}x)_0) - s^{-1/2} f(s, (e^{sA}z)_0 + (e^{tA}x)_0) \right| \right. \right. \\ &\quad \left. \left. + |\bar{\ell}_{0,n}(s, (e^{sA}z)_0 + (e^{tA}x)_0) - \bar{\ell}_0(s, (e^{sA}z)_0 + (e^{tA}x)_0)| \right)^2 \mathcal{N}(0, Q_{t-s})(dz) \right)^{1/2} ds. \end{aligned}$$

Applying the Dominated Convergence Theorem as for the proof of (4.8) we get the final claim

$$\sup_{x \in (0, T] \times \mathcal{K}} |t^{1/2} \nabla^B v_n(t, x) - t^{1/2} \nabla^B v(t, x)| \rightarrow 0, \quad \text{for any compact set } \mathcal{K} \subset \mathcal{H}.$$

□

Notice that, using the terminology of [7, 20], the above result implies that a mild solution (4.1) is also a \mathcal{K} -strong solution. In general, in an infinite dimensional Hilbert space H , existence of \mathcal{K} -strong solutions is not a routine application of the theory of evolution equations, as the operator \mathcal{L} formally introduced in 5.14 is not the infinitesimal generator of a strongly continuous semigroup in the Banach space $C_b(H)$. To overcome this difficulty in the already mentioned paper [7] the theory of weakly continuous (or \mathcal{K} -continuous) semigroups has been used.

Remark 4.4 *The approximation results proved just above is needed to prove the fundamental identity, (see next Proposition 5.1) which is the key point to get the verification theorem and the existence of optimal feedback controls. The idea is to apply Ito's formula to the approximating sequence w_n composed with the state process Y and then to pass to the limit for $n \rightarrow +\infty$ (see e.g. [20] or [12, Section 4.4]). However in the literature the approximating sequence is taken more regular, i.e. the w_n are required to be classical solutions (see e.g. [10, Section 6.2, p.103]) of (4.1). This in particular means that $\nabla w_n \in D(A^*)$ and this fact is crucial since it allows to apply Ito's formula without requiring that the state process Y belongs to $D(A)$, which would be a too strong requirement.*

In our case the used approximating procedure does not give rise in general to functions w_n with $\nabla w_n \in D(A^)$. Indeed for our purposes we need that the approximants of the data $\phi, \ell_0, \mathcal{F}$ remain all in the space $\Sigma_{T,1/2}$; without this, since we only have "partial" smoothing, it is not clear at all how to prove the convergence of the derivative $\nabla^B w_n$ (which is needed when we pass to the limit to prove the fundamental identity in next subsection). Hence, in particular, since we need that, for all $n \in \mathbb{N}$, $\mathcal{F}_n \in \Sigma_{T,1/2}^1$, and since \mathcal{F} is written in terms of f , we approximate f by f_n , and this procedure gives the approximants \mathcal{F}_n of \mathcal{F} . In this way $\mathcal{F}_n \in \Sigma_{T,1/2}^1$ but $\nabla \mathcal{F}_n \notin D(A^*)$.*

Summing up, we are only able to find approximating strict solutions and not classical solutions. Since the state process Y does not belong to $D(A)$ this fact will force us to introduce suitable regularizations Y_k of it (see the proof of Proposition 5.1).

Remark 4.5 *Calling $\ell_n := \ell_{0,n} + \mathcal{F}_n - H_{\min}(\nabla^B w_n)$ is not difficult to see that the sequence w_n is a strict solution of the approximating HJB equation*

$$\begin{cases} \frac{\partial w(t, x)}{\partial t} = \frac{1}{2} \text{Tr } GG^* \nabla^2 w(t, x) + \langle Ax, \nabla w(t, x) \rangle_{\mathcal{H}} + H_{\min}(\nabla^B w(t, x)) + \ell_n(t, x), & t \in [0, T], x \in \mathcal{H}, \\ w(0, x) = \phi_n(x), \end{cases}$$

This means, with the terminology used e.g. in [20], that w is a \mathcal{K} -strong solution of (1.5). We do not go deeper into this since here we use the approximation only as a tool to solve our stochastic optimal control problem.

Remark 4.6 *It can be noticed, see also [24], that our results on the HJB equation could be extended without difficulties to the case when the boundedness assumption on $\bar{\phi}$ and $\bar{\ell}_0$, and so on ϕ and ℓ_0 , is replaced by a polynomial growth assumption: namely that, for some $N \in \mathbb{N}$, the functions*

$$x \mapsto \frac{\phi(x)}{1 + |x|^N}, \quad (t, x) \mapsto \frac{\ell_0(t, x)}{1 + |x|^N}, \quad (4.9)$$

are bounded. Also 4.3 can be easily generalized to the case when the data ϕ and ℓ_0 are not bounded but satisfy a polynomial growth condition in the variable x as from (4.9).

Moreover it is still possible to extend the results of Lemma 4.3 to the case when ϕ and ℓ_0 are only measurable. In this case the approximations would take place in the sense of the π -convergence, which is weaker than the \mathcal{K} -convergence and towards the \mathcal{K} -convergence has also the disadvantage of being not metrizable. For more on the notion of π -convergence the reader can see [10, Section 6.3] (see also [11] and [30]).

5 Verification Theorem and Optimal Feedbacks

The aim of this section is to provide a verification theorem and the existence of optimal feedback controls for our problem. This in particular will imply that the mild solution v of the HJB equation (1.5) built in Theorem 3.6 is equal to the value function V of our optimal control problem.

The main tool needed to get the wanted results is an identity (often called “*fundamental identity*”, see equation (5.1)) satisfied by the solutions of the HJB equation. When the solution is smooth enough (e.g. it belongs to $UC_b^{1,2}([0, T] \times H)$) such identity is easily proved using the Ito’s Formula. Since in our case the value function does not possess this regularity, we proceed by approximation, following the lines of Section 4. Due to the features of our problem (lack of smoothing and of the structure condition) the methods of proof used in the literature do not apply here. We will discuss the main issues along the way.

5.1 The Fundamental Identity and the Verification Theorem

Now we finally go back to the control problem of Section 2.2. We rewrite here for the reader convenience the state equation (2.2),

$$\begin{cases} dy(s) = a_0 y(s) ds + b_0 u(s) ds + \int_{-d}^0 b_1(\xi) u(s + \xi) d\xi + \sigma dW_s, & s \in [t, T] \\ y(t) = y_0, \\ u(\xi) = u_0(\xi), & \xi \in [-d, 0), \end{cases}$$

and its abstract reformulation (2.11),

$$\begin{cases} dY(s) = AY(s) ds + Bu(s) ds + GdW_s, & s \in [t, T] \\ Y(t) = x = (x_0, x_1). \end{cases}$$

Similarly the cost functional in (2.13) is

$$J(t, x, u) = \mathbb{E} \int_t^T (\bar{\ell}_0(s, x(s)) + \ell_1(u(s))) ds + \mathbb{E} \bar{\phi}(x(T))$$

and is rewritten as (see (2.14))

$$J(t, x; u) = \mathbb{E} \int_t^T (\ell_0(s, Y(s)) + \ell_1(u(s))) ds + \mathbb{E} \phi(Y(T)).$$

We notice that throughout this subsection and the following one, in order to avoid further technical difficulties, we keep the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ fixed. Nothing would change if we work in the weak formulation, where the probability space can change (see e.g. [32][Chapter 2] and [12][Chapter 2] for more on strong and weak formulations in finite and infinite dimension, respectively). We first prove the fundamental identity.

Proposition 5.1 *Let Hypotheses 2.3 and 2.4 hold. Let also (3.9) or (3.10) hold. Let v be the mild solution of the HJB equation (1.5) according to Definition 3.5. Then for every $t \in [0, T]$ and $x \in H$, and for every admissible control u , we have the fundamental identity*

$$v(t, x) = J(t, x; u) + \mathbb{E} \int_t^T [H_{min}(\nabla^B v(s, Y(s))) - H_{CV}(\nabla^B v(s, Y(s)); u(s))] ds. \quad (5.1)$$

Proof. Take any admissible state-control couple $(Y(\cdot), u(\cdot))$, and let $v_n(t, x) := w_n(T - t, x)$ where $(w_n)_n$ is the approximating sequence of strict solutions defined in Lemma 4.3. We want to apply the Ito formula to $v_n(t, Y(t))$. Unfortunately, as mentioned in Remark 4.4, this is not possible since the process $Y(t)$ does not live in $D(A)$. So we approximate it as follows. Set, for $k \in \mathbb{N}$, sufficiently large,

$$Y_k(s; t, x) = k(k - A)^{-1} Y(s; t, x). \quad (5.2)$$

The process Y_k is in $D(A)$, it converges to Y (\mathbb{P} -a.s. and $s \in [t, T]$ a.e.) and it is a strong solution⁵ of the Cauchy problem

$$\begin{cases} dY_k(s) = AY_k(s) ds + B_k u(s) ds + G_k dW_s, & s \in [t, T] \\ Y_k(t) = x_k, \end{cases}$$

⁵Here we mean strong in the probabilistic sense and also in the sense of [9], Section 5.6.

where $B_k = k(k - A)^{-1}B$, $G_k = k(k - A)^{-1}G$ and $x_k = k(k - A)^{-1}x$. Now observe that, thanks to (2.6) and (2.9), the operator B_k is continuous, hence we can apply Dynkin's formula (see e.g. [12, Section 1.7] or [9, Section 4.5]) to $v_n(s, Y_k(s))$ in the interval $[t, T]$, getting

$$\begin{aligned} & \mathbb{E}v_n(Y_k(T)) - v_n(t, x_k) \\ &= \mathbb{E} \int_t^T \left[v_{nt}(s, Y_k(s)) + \frac{1}{2}Tr \, GG^* \nabla^2 v(s, Y_k(s)) + \langle AY_k(s) + B_k u(s), \nabla v_n(s, Y_k(s)) \rangle_{\mathcal{H}} \right] ds. \end{aligned} \quad (5.3)$$

Using the Kolmogorov equation (4.1), whose strict solution is w_n , we then write

$$\mathbb{E}\phi_n(Y_k(T)) - v_n(t, x_k) = \mathbb{E} \int_t^T [\mathcal{F}_n(s, Y_k(s)) + \ell_{0,n}(s, Y_k(s)) + \langle B_k u(s), \nabla v_n(s, Y_k(s)) \rangle_{\mathbb{R}^m}] ds \quad (5.4)$$

We first let $k \rightarrow \infty$ in (5.4). Since $\ell_{0,n}$ and ∇v_n are bounded functions and since $\mathcal{F}_n(s, x)$ has a singularity of type $s^{-1/2}$ with respect to time and is bounded with respect to x , we can apply the Dominated Convergence Theorem to all terms but the last getting

$$\begin{aligned} & \mathbb{E}\phi_n(Y(T)) - v_n(t, x) \\ &= \mathbb{E} \int_t^T [\mathcal{F}_n(s, Y(s)) + \ell_{0,n}(s, Y(s))] ds + \lim_{k \rightarrow +\infty} \mathbb{E} \int_t^T \langle B_k u(s), \nabla v_n(s, Y_k(s)) \rangle_{\mathcal{H}} ds. \end{aligned} \quad (5.5)$$

Concerning the last term we observe first that, by (2.6) and (2.9), as well as by property of the operators $k(k - A)^{-1}$, we have that for every $u \in \mathbb{R}^m$

$$B_k u \rightarrow B u.$$

This in particular implies, by the Banach-Steinhaus Theorem, that $\{B_k u\}_k$ is uniformly bounded in \mathcal{H} . Now we use the fact that $\nabla v_n(s, x) \in \mathcal{H}$ (see (3.14)) to rewrite the integrand of the last term of (5.5) as

$$\langle B_k u(s), \nabla v_n(s, Y_k(s)) - \nabla v_n(s, Y(s)) \rangle_{\mathcal{H}} + \langle B_k u(s), \nabla v_n(s, Y(s)) \rangle_{\mathcal{H}}$$

Thanks to what said above the first term goes to 0 as $k \rightarrow +\infty$ and is dominated while the second term is also dominated and converges to $\langle B u(s), \nabla v_n(s, Y(s)) \rangle_{\mathcal{H}}$ which, thanks to (3.14), is equal to $\langle u(s), \nabla^B v_n(s, Y(s)) \rangle_{\mathbb{R}^m}$ (both convergences are clearly \mathbb{P} -a.s. and $s \in [t, T]$ a.e.). Hence

$$\lim_{k \rightarrow +\infty} \mathbb{E} \int_t^T \langle B_k u(s), \nabla v_n(s, Y_k(s)) \rangle_{\mathcal{H}} ds = \mathbb{E} \int_t^T \langle u(s), \nabla^B v_n(s, Y(s)) \rangle_{\mathbb{R}^m} ds$$

Now we let $n \rightarrow \infty$. By Lemma 4.3, we know that

$$v_n(t, x) \rightarrow v(t, x) \quad \text{and} \quad (T - t)^{1/2} \nabla^B v_n(t, x) \rightarrow (T - t)^{1/2} \nabla^B v(t, x)$$

pointwise. Moreover $v_n(t, x)$, $(T - t)^{1/2} \nabla^B v_n(t, x)$ are uniformly bounded, so that, by dominated convergence, we get

$$\mathbb{E} \int_t^T \langle u(s), \nabla^B v_n(s, Y(s)) \rangle_{\mathbb{R}^m} ds \rightarrow \mathbb{E} \int_t^T \langle u(s), \nabla^B v(s, Y(s)) \rangle_{\mathbb{R}^m} ds.$$

The convergence

$$\mathbb{E}\phi_n(x(T)) - \mathbb{E} \int_t^T [\mathcal{F}_n(s, Y(s)) + \ell_{0,n}(s, Y(s))] ds \rightarrow \mathbb{E}\phi(x(T)) - \mathbb{E} \int_t^T [H_{\min}(\nabla^B v(s, Y(s))) + \ell_0(s, Y(s))] ds$$

follows directly by the construction of the approximating sequences $(\phi_n)_n$, $(\mathcal{F}_n)_n$ and $(\ell_{0,n})_n$. Then, adding and subtracting $\mathbb{E} \int_t^T \ell_1(u(s)) ds$ and letting $n \rightarrow \infty$ in (5.5) we obtain

$$\begin{aligned} v(t, x) &= \mathbb{E}\phi(Y(T)) + \mathbb{E} \int_t^T [\ell_0(s, Y(s)) + \ell_1(u(s))] ds \\ &\quad + \mathbb{E} \int_t^T [H_{\min}(\nabla^B v(s, Y(s))) - H_{CV}(\nabla^B v(s, Y(s)); u(s))] ds \end{aligned}$$

which immediately gives the claim. \square

Remark 5.2 *One may wonder why we approximate the process Y with Y_k as in (5.2) instead of using the Yosida approximants A_k of A as, e.g., in [10, p.144]. The reason is that we need that Y_k belongs to $D(A)$, which is not guaranteed if we use Yosida approximants. A similar procedure is used, in a different context, in the book [9], in the proof of Theorem 7.7, p. 203.*

We can now pass to prove our Verification Theorem i.e. a sufficient condition of optimality given in term of the mild solution v of the HJB equation (1.5).

Theorem 5.3 *Let Hypotheses 2.3 and 2.4 hold. Let also (3.9) or (3.10) hold. Let v be the mild solution of the HJB equation (1.5) whose existence and uniqueness is proved in (3.6). Then the following holds.*

- *For all $(t, x) \in [0, T] \times \mathcal{H}$ we have $v(t, x) \leq V(t, x)$, where V is the value function defined in (2.18).*
- *Let $t \in [0, T]$ and $x \in \mathcal{H}$ be fixed. If, for an admissible control u^* , we have, calling Y^* the corresponding state,*

$$u^*(s) \in \arg \min_{u \in U} H_{CV}(\nabla^B v(s, Y^*(s); u)$$

for a.e. $s \in [t, T]$, \mathbb{P} -a.s., then the pair (u^, Y^*) is optimal for the control problem starting from x at time t and $v(t, x) = V(t, x) = J(t, x; u^*)$.*

Proof. The first statement follows directly by (5.1) due to the negativity of the integrand. Concerning the second statement, we immediately see that, when $u = u^*$ (5.1) becomes $v(t, x) = J(t, x; u^*)$. Since we know that for any admissible control u

$$J(t, x; u) \geq V(t, x) \geq v(t, x),$$

the claim immediately follows. \square

5.2 Optimal feedback controls and $v = V$

We now prove the existence of optimal feedback controls. Under the Hypotheses of Theorem 5.3 we define, for $(s, y) \in [0, T] \times \mathcal{H}$, the *feedback map*

$$\Psi(s, y) := \arg \min_{u \in U} H_{CV}(\nabla^B v(s, y); u), \quad (5.6)$$

where, as usual, v is the solution of the HJB equation (1.5). Given any $(t, x) \in [0, T] \times \mathcal{H}$, the so-called Closed Loop Equation (which here is, in general, an inclusion) is written, formally, as

$$\begin{cases} dY(s) \in AY(s)ds + B\Psi(s, Y(s))ds + GdW_s, & s \in [t, T] \\ Y(t) = x. \end{cases} \quad (5.7)$$

First of all we have the following straightforward corollary whose proof is immediate from Theorem 5.3.

Corollary 5.4 *Let the assumptions of Theorem 5.3 hold true. Let v be the mild solution of (2.20). Fix $(t, x) \in [0, T] \times H$ and assume that, on $[t, T] \times H$, the map Ψ defined in (5.6) admits a measurable selection $\psi : [t, T] \times H \rightarrow \Lambda$ such that the Closed Loop Equation*

$$\begin{cases} dY(s) = AY(s)ds + B\psi(s, Y(s))ds + GdW_s, & s \in [t, T] \\ Y(t) = x. \end{cases} \quad (5.8)$$

has a mild solution $Y_\psi(\cdot; t, x)$ (in the sense of [9, p.187]). Define, for $s \in [t, T]$, $u_\psi(s) = \psi(s, Y_\psi(s; t, x))$. Then the couple $(u_\psi(\cdot), Y_\psi(\cdot; t, x))$ is optimal at (t, x) and $v(t, x) = V(t, x)$. If, finally, $\Psi(t, x)$ is always a singleton and the mild solution of (5.8) is unique, then the optimal control is unique.

We now give sufficient conditions to verify the assumptions of Corollary 5.4. First of all define

$$\Gamma(p) := \{u \in U : \langle p, u \rangle + \ell_1(u) = H_{\min}(p)\}. \quad (5.9)$$

Then, clearly, we have $\Psi(t, x) = \Gamma(\nabla^B v(t, x))$. Observe that, under mild additional conditions on U and ℓ_1 (for example taking U compact or ℓ_1 of superlinear growth), the set Γ is nonempty for all $p \in \mathbb{R}^m$. If this is the case then, by [1, Theorems 8.2.10 and 8.2.11, Γ admits a measurable selection, i.e. there

exists a measurable function $\gamma : \mathbb{R}^m \rightarrow U$ with $\gamma(z) \in \Gamma(z)$ for every $z \in \mathbb{R}^m$. Since H_{min} is Lipschitz continuous, then Γ , and so γ , must be uniformly bounded. In some cases studied in the literature this is enough to find an optimal feedback but not in our case (read on this the subsequent Remark 5.6-(ii)). Hence to prove existence of a mild solution of the closed loop equation (5.8), as requested in Corollary 5.4, we need more regularity of the feedback term $\psi(s, y) = \gamma(\nabla^B v(s, y))$. Beyond the smooth assumptions on the coefficients required in the second part of Theorem 3.6, which give the regularity of $\nabla^B v(t, x)$, we need the following assumption about the map Γ .

Hypothesis 5.5 *The set-valued map Γ defined in (5.9) is always non empty; moreover it admits a Lipschitz continuous selection γ .*

Remark 5.6 (i) *Notice that the above Hypothesis is verified if we assume that $\ell_1 : \mathbb{R}^m \rightarrow \mathbb{R}$ is differentiable with an invertible derivative and that $(\ell'_1)^{-1}$ is Lipschitz continuous. Indeed in this case, see (2.19), the infimum of H_{CV} is achieved at*

$$u = (\ell'_1)^{-1}(z), \text{ so that } \Gamma_0(z) = (\ell'_1)^{-1}(z).$$

(ii) *The problem of the lack of regularity of the feedback law is sometimes faced (see e.g. in [17]) by formulating the optimal control problem in the weak sense (see e.g. [16] or [32], Section 4.2) and then using Girsanov Theorem to prove existence, in the weak sense, of a mild solution of (5.8) when the map ψ is only measurable and bounded. This is not possible here due to the already mentioned absence of the structure condition in the controlled state equation (i.e. the control affects the system not only through the noise).*

Taking the selection γ from Hypothesis 5.5 we consider the closed loop equation

$$\begin{cases} dY(s) = AY(s)ds + B\gamma(\nabla^B v(s, Y(s)))ds + GdW_s, & s \in [t, T] \\ Y(t) = x = (x_0, x_1), \end{cases} \quad (5.10)$$

and we have the following result.

Theorem 5.7 *Assume that Hypotheses 2.3, 2.4, 2.5 and 5.5 hold true. Fix any $(t, x) \in [0, T) \times H$. Let also (3.9) or (3.10) hold. Then the closed loop equation (5.10) admits a unique mild solution $Y_\gamma(\cdot; t, x)$ (in the sense of [9, p.187]) and setting*

$$u_\gamma(s) = \gamma(\nabla^B v(s, Y_\gamma(s; t, x))), \quad s \in [t, T]$$

we obtain an optimal control at (t, x) . Moreover $v(t, x) = V(t, x)$.

Proof. Thanks to Corollary 5.4 it is enough to prove the existence and uniqueness of the mild solution of (5.10). We apply a fixed point theorem to the following integral form of (5.10):

$$Y(s) = e^{(s-t)A}x + \int_t^s e^{(s-r)A}G dW_r + \int_t^s e^{(s-r)A}B\gamma(\nabla^B v(r, \overline{X}_r))dr. \quad (5.11)$$

By Hypothesis 2.5 and the second part of Theorem 3.6 we get that the mild solution v of the HJB equation (2.20) is differentiable, with bounded derivative. Moreover, since, again by the second part of Theorem 3.6, v admits the second order derivative $\nabla^B \nabla v$, and $\nabla^B \nabla v = \nabla \nabla^B v$, we deduce that $t^{1/2} \nabla^B v(t, \cdot)$ is Lipschitz continuous, uniformly with respect to t . Using this Lipschitz property we can solve (5.11) by a fixed point argument. Since the argument to do this is straightforward we only show how to estimate the difficult term in (5.11). We have

$$\begin{aligned} \int_t^s |e^{(s-r)A}B(\gamma(\nabla^B v(r, \overline{X}(r))) - \gamma(\nabla^B v(r, \overline{Y}(r))))|_{\mathcal{H}} dr &\leq \int_t^s C|\gamma(\nabla^B v(r, \overline{X}(r))) - \gamma(\nabla^B v(r, \overline{Y}(r)))|_{\mathbb{R}^m} dr \\ &\leq C \int_t^s |\nabla^B v(r, \overline{X}(r)) - \nabla^B v(r, \overline{Y}(r))|_{\mathbb{R}^m} dr \leq C \int_t^s r^{-1/2} |\overline{X}(r) - \overline{Y}(r)|_{\mathcal{H}} dr \end{aligned}$$

where C is a constant that can change its values from line to line. \square

We devote our final result to show that the identification $v = V$ can be done, using an approximation procedure, also in cases when we do not know if optimal feedback controls exist.

Theorem 5.8 *Let Hypotheses 2.3, 2.4 hold. Let also (3.9) or (3.10) hold. Moreover let Hypotheses 2.5-(ii) and 5.5 hold and let ϕ and ℓ_0 be uniformly continuous. Then $v = V$.*

Proof. We approximate ℓ_0 and ϕ by approximating $\bar{\ell}_0$ and $\bar{\phi}$ with standard approximants $\bar{\ell}_{0,n}$ and $\bar{\phi}_n$ built by convolutions. We set

$$J_n(t, x; u) = \mathbb{E} \int_t^T (\ell_{0,n}(s, Y(s)) + \ell_1(u(s))) ds + \mathbb{E} \phi_n(Y(T)) \quad (5.12)$$

and call w_n the mild solution of the HJB equation

$$\begin{cases} \frac{\partial w(t, x)}{\partial t} = \mathcal{L}[w(t, \cdot)](x) + \ell_{0,n}(t, x) + H_{\min}(\nabla^B w(t, x)), & t \in [0, T], x \in \mathcal{H}, \\ w(0, x) = \phi_n(x), \end{cases} \quad (5.13)$$

where \mathcal{L} is given by

$$\mathcal{L}[f](x) = \frac{1}{2} \text{Tr} GG^* \nabla^2 f(x) + \langle x, A^* \nabla f(x) \rangle. \quad (5.14)$$

In particular w_n satisfies the integral equation

$$w_n(t, x) = R_t[\phi_n](x) + \int_0^t R_{t-s}[H_{\min}(\nabla^B w_n(s, \cdot) + \ell_{0,n}(s, \cdot))](x) ds, \quad t \in [0, T], x \in \mathcal{H}, \quad (5.15)$$

By Theorem 5.7 calling $v_n(t, x) = w_n(T - t, x)$ we have

$$v_n(t, x) = V_n(t, x) := \inf_{u \in \mathcal{U}} J_n(t, x; u). \quad (5.16)$$

and there exists an optimal feedback control $u_n(s) = \psi_n(s, Y(s))$. Moreover, by Lemma 4.3 we know that

$$v_n(t, x) \xrightarrow{K} v(t, x).$$

Now it is enough to prove that $V_n(t, x) \rightarrow V(t, x)$ pointwise. Given $\varepsilon > 0$, we have, for n large enough,

$$\begin{aligned} V_n(t, x) &= \inf_{u \in \mathcal{U}} \left[\mathbb{E} \int_t^T (\ell_{0,n}(s, Y(s)) + \ell_1(u(s))) ds + \mathbb{E} \phi_n(Y(T)) \right] \\ &= \inf_{u \in \mathcal{U}} \left[\mathbb{E} \int_t^T (\ell_0(s, Y(s)) + \ell_1(u(s))) ds + \mathbb{E} \phi(Y(T)) \right. \\ &\quad \left. + \mathbb{E} \int_t^T (\ell_{0,n}(s, Y(s)) - \ell_0(s, Y(s))) ds + \mathbb{E} [\phi_n(Y(T)) - \phi(Y(T))] \right] \\ &\leq \inf_{u \in \mathcal{U}} \left[\mathbb{E} \int_t^T (\ell_0(s, Y(s)) + \ell_1(u(s))) ds + \mathbb{E} \phi(Y(T)) \right] + \varepsilon, \end{aligned}$$

where the last passage follows by the Dominated Convergence Theorem, and since ϕ and $\ell_{0,n}$ are uniformly continuous. We have shown that

$$V_n(t, x) \leq V(t, x) + \varepsilon.$$

Exchanging the role of V_n and V we also find that the reverse inequality holds true. Hence $V_n \rightarrow V$ pointwise and the claim follows. \square

Remark 5.9 *In Theorem 5.8 we have assumed further uniform continuity on the data. When U is compact the result still remain true if the data are only continuous.*

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